Correspondence

Wave Propagation in Coaxial-Cylindrical Slow-Wave Systems*

Introduction

Recent interest in E-type traveling-wave tubes,[1] in which a ribbon-shaped electron beam is caused to follow a circular path by balancing the centrifugal force of the particle against a steady radial electric field force, has led to an investigation of wave propagation in azimuthally reentrant and nonreentrant coaxial-cylindrical slow-wave structures. The study is facilitated by the simplifying approximation that the actual azimuthally-periodic slow-wave circuit, situated along the inner conductor, can be replaced by a smooth dielectric cylinder. While such a dielectric cylinder would probably not be employed in the construction of an actual tube, because of its small value of surface impedance, it serves as a convenient model in determining the general forms of functional dependence for the field equations.

The Field Equations

Although the first exhaustive study of wave propagation in azimuthally reentrant coaxial-cylindrical transmission systems appears to have been given by Kalahne,[2] in 1905, the problem has since been widely treated.[3-5] In addition, Buchholz[6] has studied the related problem of wave propagation around a circular bend in a rectangular waveguide, and Waldron[7] has investigated the characteristics of wave propagation in helical waveguides of square cross section.

When the analysis is extended to the case involving periodic structures situated along the center conductor to produce azimuthally slow waves, the boundary conditions are altered to include the periodic variation of the inner conductor radius with the spatial angle θ. This change of shape leads to an infinite number of azimuthal space harmonic components, each one of which is characterized by a particular circular propagation constant. The amplitudes and phases of these harmonic components are determined by satisfying the conditions imposed on the fields by the periodic boundaries of the circuit.

Though the procedure can, in principle, be applied to any azimuthally-periodic boundary, the method becomes involved for any but the simplest structures. Moreover, each time the geometry is changed the evaluation must be repeated, so that practical considerations tend to restrict the analysis to a small number of potentially important configurations. Despite these difficulties, certain general properties of azimuthally periodic slow-wave structures can be inferred from a study of typical situations. Thus, when the circuit is reentrant, or when the spatial period of the structure is such that the fields would be undisturbed by making it reentrant, the azimuthal space harmonics form a countably infinite set of integral values. When the nature of the structure is such that the fields would be disturbed by making it reentrant, the space harmonics form a countably infinite set of rational fractional, or general nonintegral, order values. Either of these situations involves a discrete summation over the associated circular propagation constants in order to obtain the field expressions. It follows that the general expressions for the electric and magnetic field intensities have the form

$$E_z = \mathrm{Re} \sum_{n,\beta_0} A_{n\beta_0} V_{\beta_0}(k_n r) \cos\left(\frac{n\pi}{L} z\right) e^{j(\omega t - \beta_0 \theta)}, \tag{1}$$

$$H_z = \mathrm{Re} \sum_{n,\beta_0} jB_{n\beta_0} Y_{\beta_0}(k_n r) \sin\left(\frac{n\pi}{L} z\right) e^{j(\omega t - \beta_0 \theta)}, \tag{2}$$

$$E_r = \mathrm{Re} \sum_{n,\beta_0} \left\{ A_{n\beta_0}\left(\frac{-n\pi}{k_n L}\right) V_{\beta_0}'(k_n r) + B_{n\beta_0}\left(\frac{-j\omega\mu_0\beta_0}{k_n^2 r}\right) Y_{\beta_0}(k_n r) \right\} \sin\left(\frac{n\pi}{L} z\right) e^{j(\omega t - \beta_0 \theta)}, \tag{3}$$

$$E_\theta = \mathrm{Re} \sum_{n,\beta_0} \left\{ A_{n\beta_0}\left(\frac{jn\pi\beta_0}{k_n^2 r L}\right) V_{\beta_0}(k_n r) + B_{n\beta_0}\left(\frac{-\omega\mu_0}{k_n}\right) Y_{\beta_0}'(k_n r) \right\} \sin\left(\frac{n\pi}{L} z\right) e^{j(\omega t - \beta_0 \theta)}, \tag{4}$$

$$H_r = \mathrm{Re} \sum_{n,\beta_0} \left\{ A_{n\beta_0}\left(\frac{\omega\epsilon_0\beta_0}{k_n^2 r}\right) V_{\beta_0}(k_n r) + B_{n\beta_0}\left(\frac{jn\pi}{k_n L}\right) Y_{\beta_0}'(k_n r) \right\} \cos\left(\frac{n\pi}{L} z\right) e^{j(\omega t - \beta_0 \theta)}, \tag{5}$$

$$H_\theta = \sum_{n,\beta_0} \left\{ A_{n\beta_0}\left(\frac{-j\omega\epsilon_0}{k_n}\right) V_{\beta_0}'(k_n r) + B_{n\beta_0}\left(\frac{n\pi\beta_0}{k_n^2 r L}\right) Y_{\beta_0}(k_n r) \right\} \cos\left(\frac{n\pi}{L} z\right) e^{j(\omega t - \beta_0 \theta)}, \tag{6}$$

where

μ_0 = free-space permeability in henries per meter,

ϵ_0 = free-space permittivity in farads per meter,

β_0 = circular propagation constant in electrical radians per spatial radian,

ω = electrical angular velocity of the RF wave in electrical radians per second,

L = height of the waveguide parallel to the cylindrical axis in meters,

r = radial coordinate variable in meters,

θ = azimuthal coordinate variable in radians,

z = axial coordinate variable in meters,

t = time in seconds,

$k_n = \sqrt{(n\pi/L)^2 - \omega^2\mu_0\epsilon_0}$ electrical radians per meter,

$A_{n\beta_0}$ = a quantity depending on β_0 and n, which is a measure of the space harmonic amplitude of the associated E-mode wave.

$B_{n\beta_0}$ = a quantity depending on β_0 and n, which is a measure of the space harmonic amplitude of the associated H-mode wave,

$V_{\beta_0}(k_n r) = I_{\beta_0}(k_n r) K_{\beta_0}(k_n r_s) - K_{\beta_0}(k_n r) I_{\beta_0}(k_n r_s)$,

$Y_{\beta_0}(k_n r) = I_{\beta_0}(k_n r) K_{\beta_0}'(k_n r_s) - K_{\beta_0}(k_n r) I_{\beta_0}'(k_n r_s)$,

$I_{\beta_0}(k_n r)$, $K_{\beta_0}(k_n r)$ = hyperbolic Bessel functions of the first and second kinds, respectively, of order β_0 and argument $k_n r$,

r_s = radius of the inside surface of the outer conductor (in an electron beam device this corresponds to the sole radius) in meters.

The primes appearing in the foregoing expression indicate differentiation of the associated functions with respect to $k_n r$. The terminology of Morse and Feshback[8] is used in referring to the Bessel functions as "hyperbolic" rather than "modified," since these expressions are related to the corresponding Bessel functions of real argument in a manner similar to that of the more elementary trigonometric and hyperbolic functions.

The transition from the Bessel functions of real argument, which accompany higher-mode *propagating* waves in coaxial-cylindrical transmission systems, to the hyperbolic Bessel functions employed here results from the premise that the fringing circuit fields, which permeate the interaction space, could not exist in the absence of the slow-wave circuit. The introduction of a lossless dielectric cylinder surrounding the center con-

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[4] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1941.

[5] N. Marcuvitz, "Waveguide Handbook," MIT Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 10; 1951.

[6] H. Buchholz, "Effect of curvature of rectangular hollow conductors on the phase constant of ultra-short waves," *Elektrische Nachrichtentech.*, vol. 16, pp. 73–85; March, 1939.

[7] R. A. Waldron, "Theory of the helical waveguide of rectangular cross-section," *J. Brit. IRE*, vol. 17, pp. 577–592; October, 1957.

[8] P. M. Morse and H. Feshback, "Methods of Theoretical Physics," McGraw-Hill Book Co., Inc., New York, N. Y., pts. 1, 2; 1953.

ductor (which is assumed to be homogeneous and isotropic), as a replacement for the actual slow-wave circuit, implies the presence of both axial and azimuthal slowing, although only the latter is of importance in E-type devices. A limitation inherent in the use of the dielectric cylinder arises from the fact that the dielectric is perfectly smooth, so that analyses based on this model indicate the presence of only one of the infinite number of possible space harmonics produced in the actual structure. However, the constraint is not serious so long as the conditions are such that the electron beam effectively interacts with only one space-harmonic component of the RF field.

A Further Examination of the Field Relations

A study of the field equations leads to some interesting conclusions concerning the characteristics of the transmission system described above. It is convenient in this study to regard the entire cross-sectional area of the system as being divided into the "slow-wave region," containing the periodic RF structure, and the "interaction space" which surrounds the slow-wave region. Thus, in the slow-wave region the radial propagation constant k_{2n} is given by

$$k_{2n}^2 = \left[\omega^2\mu_0\epsilon_2 - \left(\frac{n\pi}{L}\right)^2\right], \tag{7}$$

where ϵ_2 is the permittivity associated with the dielectric cylinder in farads per meter. It is apparent that k_{2n}^2 must be greater than zero, for otherwise wave propagation would be cut off in the slow-wave region, and therefore in the entire system. It follows that the radial variation of all field components in the slow-wave region must exhibit the real argument (rather than the imaginary argument) Bessel function behavior. The cutoff condition for axial propagation in the slow-wave region, $k_{2n}^2 \equiv 0$, leads to

$$|n| \leqq \frac{\omega L}{\pi}\sqrt{\mu_0\epsilon_2}, \tag{8}$$

as shown in Fig. 1. It may be noted that the allowed range of axial eigenvalues, corresponding to the integer n values, increases with frequency, with the height of the waveguide parallel to the cylindrical axis, and with the dielectric constant.

Since only the cutoff modes were assumed to be present in deriving the field relations given above, then the radial propagation constant appropriate to the interaction space k_n yields, for the condition $k_n^2=0$,

$$\left|\frac{\omega L}{\pi}\sqrt{\mu_0\epsilon_0}\right| < |n| < \infty,$$

as shown in Fig. 2. Thus, a finite and bounded set of axial eigenvalues associated with wave propagation in the slow-wave region leads to a pair of sets of infinite eigenvalues associated with the cutoff modes permeating the interaction region. It is apparent, however, that if the guide height L or the operating frequency is sufficiently increased to permit E- or H-wave *propagating* modes to exist in this space, then the elements of the periodic structure act as sources of radiation that excite strong fields in the region surrounding the RF circuit. The *propagating-wave* axial eigenvalues then

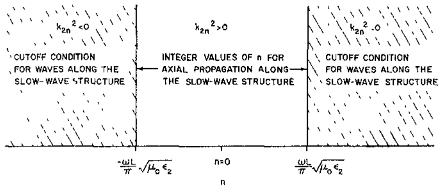

Fig. 1—Graphical representation of the range of n in the slow-wave region.

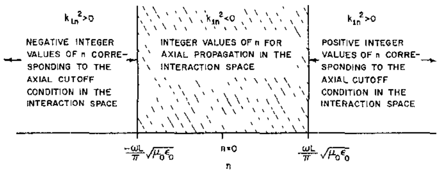

Fig. 2—Graphical representation of the range of n in the interaction space.

lie in the shaded area of Fig. 2, and the radial variation of all field components now take on the real-argument Bessel function variation.

Since the Hankel functions of the first and second kinds are linearly related to the two types of Bessel functions of real argument, it may be shown that radially propagating waves accompany axially propagating waves, and that radially attenuating functions accompany axially cutoff modes. The fields on the slow-wave structure must always be associated with radially and axially propagating modes, since operation below the cutoff frequency of the RF circuit suppresses electromagnetic radiation in the entire system. The fields in the interaction space may be associated either with radially and axially propagating or attenuating waves, depending on the relation of the operating frequency to the cutoff frequency of the interaction space.

The lowest E-mode field occurs for $n=0$, because E_z has the axial variation

$$\cos\left(\frac{n\pi}{L}z\right),$$

while the lowest H-mode field occurs for $n=1$, because H_z has the axial variation

$$\sin\left(\frac{n\pi}{L}z\right).$$

However, since E_θ possesses the same axial variation as H_z, it follows that the lowest mode, for which effective electron-wave interaction in E-type devices is possible, occurs when $n=1$.

In contrast to the "permitted sets" of axial eigenvalues, all field components possess an azimuthal variation of the form $e-j\beta_0\theta$. A study of this function leads to the interesting physical intepretation that, as far as azimuthal variations are concerned, the waves may be regarded as propagating in a Riemann space.[9] Each *spatial* excursion of θ corresponding to 2π electrical radians of $\beta_0\theta$ causes the waves to encounter a "new leaf" of this surface, so that the total number of leaves required to provide a closed domain of electrical angular values of the fields is equal to the circular propagation constant. Thus, β_0 is a measure of the number of leaves encompassed in completing a spatial excursion of 2π radians. The order of the real and hyperbolic Bessel functions which specify the radial behavior of all fields are therefore determined by the number of Riemann leaves involved.

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Ring Network Filter*

A filter circuit comprising two ring networks which are connected by two quarter-wavelength lines is described. This circuit has one output with a relatively wide pass band and a second output with a sharp rejection band. A printed microstrip version designed for 1-Gc operation is shown in Fig. 1. Input is at terminal 1, band-pass output at terminal 3, and rejection band output at terminal 2.

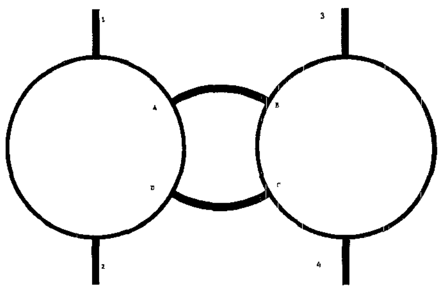

Fig. 1—1-Gc filter on $\frac{1}{16}$- inch teflon fiberglass.

A heuristic explanation of circuit action at the center frequency is as follows: Current entering arm 1 tends to divide equally between terminals A and 2 of the first ring network. A portion of the current entering terminal A travels around the loop $ABCD$, arriving at terminal D. Here one-half of the current goes to terminal 2 and one-half to terminal A. The part going to terminal A arrives in-phase while the part going to terminal 2 arrives antiphase with the currents flowing directly from terminal 1. The resultant partial cancellation at terminal 2 forces more current to enter arm AB, which in turn reduces the output from terminal 2 even more. Equilibrium is reached when the net current at terminal 2 is zero and all current into terminal 1 leaves by terminal 3. For zero circuit losses, cancellation of currents at terminal 2 would be complete, and there would be zero insertion loss between terminal 1 and terminal 3. Rejection occurs for a narrow band of frequencies at or near which the loop $ABCDA$ is one-wavelength long.

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